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DESIGN OF LAYERED VISCOELASTIC SHELLS FROM A DISCRETE SET OF MATERIALS

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The current state of the problem of optimal design of complex structures from a specified set of materials under various restrictions on their characteristics is described in detail in [1–3] where particular methods were proposed for the synthesis of layered elastic cylindrical shells that ensure necessary damping of vibrations under various external actions and restrictions on the mass or total thickness of a structure.

Among the problems of the dynamics of elastic structures, the problem of free vibrations occupies a special place. The reason is that the characteristics of free vibrations (natural frequencies and modes) completely determine the individual properties of a mechanical system and are most important for analysis of its forced vibrations. Therefore, the problems of structural synthesis from a finite set of materials with various restrictions on eigenfrequencies are of special interest. Alekhin's paper concerning the problem of synthesis of a layered cylinder and a sphere of minimal mass [4] has been so far the first and only paper on this subject.

The widespread use of polymeric materials in engineering necessitates the study of the problems of optimal design of inhomogeneous structures with viscoelastic properties. Therefore, it is of great interest to analyze the peculiarities of the problems of optimal design of viscoelastic systems in relation to similar problems for elastic structures. In addition, this analysis is interesting and is of significance, because the damping properties of viscoelastic materials can be used to advantage in designing structures from a material that possesses the required properties. In other words, the problem arises whether the peculiarities of wave passage through the boundaries of various materials can improve the viscoelastic structure. This question can be put in another way: can viscoelasticity be a dominating physical factor compared with the effects of reflection and refraction at the boundaries?

There are a number of papers devoted to direct problems of calculation of viscoelastic structural characteristics (see, e.g., [5]). The most important scientific result of these studies lies in the absence of a monotonic dependence of the dissipative characteristics of viscoelastic structures on the geometrical and other parameters of structural inhomogeneity. This result can serve as a basis for formulation of the problems of synthesis of layered structures from viscoelastic materials with restrictions permitting the satisfaction of structural limitations important for practical applications (minimum weight, maximum damping factor, etc.).

1. Let us analyze the effect of the arrangement of a structure synthesized from a finite set of viscoelastic materials (the viscoelastic parameters of materials and the location and thicknesses of layers) on the damping of free vibrations using the example of a multilayered spherical shell in which each layer is viscoelastic and its mechanical properties depend on the layer number n.

The problems of free vibrations are classified as problems in which the inertial terms completely determine the material's behavior. For this, it is necessary that the boundary conditions be such that the work of all external and mass forces identically equals zero.

With allowance for the viscoelastic analogy, the direct problem of natural vibrations of a viscoelastic spherical shell can be solved similarly to the corresponding problem of the elasticity theory in which elasticity moduli are replaced by complex viscoelasticity moduli.

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TABLE 1

Material number	ρ	λ	μ	ω
1	1	6	6	7.04
2	2	16	16	8.14
3	4	34	34	8.39

Free vibrations of a homogeneous viscoelastic sphere are described by the following boundary-value problem:

$$\frac{\partial \sigma_r}{\partial r} + 2\,\frac{\sigma_r - \sigma_\varphi}{r} = \rho \frac{\partial^2 u}{\partial t^2}; \tag{1.1}$$

$$\sigma_r = (\bar\lambda + 2\bar\mu)\frac{\partial u}{\partial r} + 2\bar\lambda\,\frac{u}{r}; \tag{1.2}$$

$$\sigma_\varphi = 2(\bar\lambda + \bar\mu)\,\frac{u}{r} + \bar\lambda\,\frac{\partial u}{\partial r}, \quad R_1 < r < R_2; \tag{1.3}$$

$$\sigma_r(R_1) = \sigma_r(R_2) = 0. \tag{1.4}$$

Here R_1 and R_2 are the inner and outer radii of the spherical shell and $\bar\lambda$ and $\bar\mu$ are the viscoelastic Lamé parameters which can be represented by means of the freezing method [6] as follows:

$$\bar\lambda_n = \lambda_n[1 - \Gamma^c_{\lambda n}(\omega_{\mathrm{Re}}) - i\Gamma^s_{\lambda n}(\omega_{\mathrm{Re}})], \quad \bar\mu_n = \mu_n[1 - \Gamma^c_{\mu n}(\omega_{\mathrm{Re}}) - i\Gamma^s_{\mu n}(\omega_{\mathrm{Re}})]; \tag{1.5}$$

$$\begin{gathered}\Gamma^c_{\lambda n}(\omega_{\mathrm{Re}}) = \int_0^\infty R_{\lambda n}(\tau)\cos(\omega_{\mathrm{Re}}\tau)\,d\tau, \quad \Gamma^s_{\lambda n}(\omega_{\mathrm{Re}}) = \int_0^\infty R_{\lambda n}(\tau)\sin(\omega_{\mathrm{Re}}\tau)\,d\tau,\\ \Gamma^c_{\mu n}(\omega_{\mathrm{Re}}) = \int_0^\infty R_{\mu n}(\tau)\cos(\omega_{\mathrm{Re}}\tau)\,d\tau, \quad \Gamma^s_{\mu n}(\omega_{\mathrm{Re}}) = \int_0^\infty R_{\mu n}(\tau)\sin(\omega_{\mathrm{Re}}\tau)\,d\tau,\end{gathered} \tag{1.6}$$

where λ_n, μ_n, $R_{\lambda n}$, and $R_{\mu n}$ are the Lamé parameters and the parameters of the relaxation kernels of the material that occupies the layer n (for a homogeneous sphere, $n = 1$) and ω_{Re} is a real constant.

The solution of problem (1.1)–(1.4) is of the form

$$u(r) = (C_1 + C_2)\Big\{-\frac{æ^2}{r}\cos(ær) + \frac{æ}{r^2}\sin(ær)\Big\} + i(C_1 - C_2)\Big\{-\frac{æ}{r^2}\cos(ær) - \frac{æ^2}{r}\sin(ær)\Big\}; \tag{1.7}$$

$$\begin{aligned}\sigma_r(r) = {}&(C_1 + C_2)[(4\muæ/r)\cos(ær) - (4\mu/r^2 - æ^2(\lambda + 2\mu))\sin(ær)]\,(æ/r)\\ &+ i(C_1 - C_2)[(4\muæ/r)\sin(ær) + (4\mu/r^2 - æ^2(\lambda + 2\mu))\cos(ær)]\,(æ/r).\end{aligned} \tag{1.8}$$

Here $æ^2 = \rho\omega^2/(\lambda + 2\mu)$; and the baı λ and μ is omitted.

Determining the constants from the boundary conditions (1.4), we obtain the characteristic equation

$$\begin{aligned}&\cos[æ(R_2 - R_1)]\Big\{\frac{4\muæ}{R_1}\Big(\frac{4\mu}{R_2^2} - \rho\omega^2\Big) - \frac{4\muæ}{R_2}\Big(\frac{4\mu}{R_1^2} - \rho\omega^2\Big)\Big\}\\ &+ \sin[æ(R_2 - R_1)]\Big\{\frac{(4\muæ)^2}{R_1R_2} + \Big(\frac{4\mu}{R_1^2} - \rho\omega^2\Big)\Big(\frac{4\mu}{R_2^2} - \rho\omega^2\Big)\Big\} = 0.\end{aligned} \tag{1.9}$$

For elastic materials, the coefficients λ and μ are real, which corresponds to the equalities $\bar\lambda = \lambda$ and $\bar\mu = \mu$ in relations (1.5). Using the calculation results of Alekhin [4], one can check the validity of the solution of (1.9). The initial values of the parameters are listed in Table 1. It was assumed that $R_1 = 0.8$ and $R_2 = 1.0$. The eigenfrequencies for each of the three materials presented in Table 1 agree with the results of [4] up to the third decimal point.

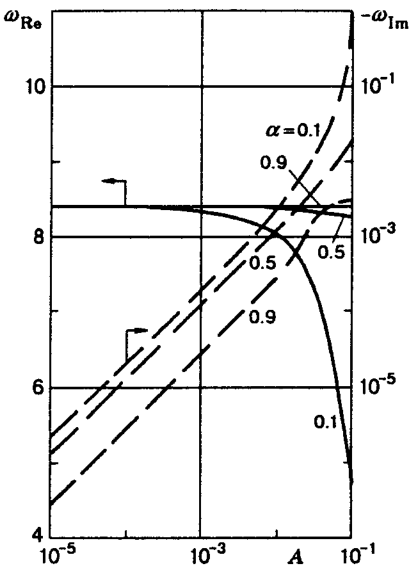

Fig. 1

In the description of viscoelastic materials, the volumetric strain was assumed to be purely elastic, i.e., the volume-compressibility modulus $K = \lambda + (2/3)\mu$ is a constant, and, to describe shearing strains, the Rzhanitsyn–Koltunov kernel [7, 8] $R_\mu = A\exp(-\beta t)/t^{1-\alpha}$ was used (in the mechanics of polymers, this kernel is mostly used).

In the computational experiment, the influence of the parameters A and α on the damping factor of natural vibrations of a homogeneous spherical viscoelastic shell was studied. The outer and inner shell radii and the β value were chosen as follows: $R_1 = 0.8$, $R_2 = 1.0$, and $\beta = 0.05$. In this study, the conclusion of [7] on the negligible influence of the parameter β on the damping factor was taken into account. This result is not universal and is valid only for materials with small β.

Figure 1 shows the results of calculations carried out using only the elasticity constants for material No. 3 in Table 1, since the general tendencies to variations in rheologic parameters are the same for all materials (solid curves refer to the real part of the complex frequency ω_{Re}, and dashed curves refer to the imaginary part of the complex frequency ω_{Im}).

Analysis of the results permits one to draw the following conclusions.

(1) For $\alpha = 0.9$, the real part of the complex frequency is not dependent on the parameter A and is equal to the natural frequency of elastic vibrations. A similar tendency is also observed for $\alpha = 0.5$ if $A < 10^{-2}$, and then ω_{Re} begins decreasing slightly.

(2) For low α, the behavior of ω_{Re} becomes more complicated: the value of ω_{Re} remains constant for $A < 10^{-3}$ and drastically decreases for other values.

(3) For all α, the imaginary part of the complex frequency grows with parameter A. It first increases according to a linear law, the linear low is then violated, and deviation from linearity occurs, depending on the values of α and of the elasticity constants: for $\alpha = 0.1$, the rate of ω_{Im} growth increases, while, for $\alpha = 0.9$, the rate first grows slightly and then decreases until a horizontal asymptote is reached (see, for example, the curve for $\alpha = 0.9$). At the same time, for $\alpha = 0.5$ the ω_{Im}–A dependence remains linear almost over the entire range of the parameters.

2. The solutions of (1.7) and (1.8) can obviously be extended to a multilayered sphere [9]. We introduce the following notation:

$$\Delta_i = (c_i(r_{i-1})b_i(r_{i-1}) - d_i(r_{i-1})a_i(r_{i-1}))^{-1} \quad (i = \overline{1, N}),$$

$$a_i(r_{i-1}) = -\frac{æ_i^2}{r_{i-1}}\cos(æ_i r_{i-1}) + \frac{æ_i}{r_{i-1}^2}\sin(æ_i r_{i-1}),$$

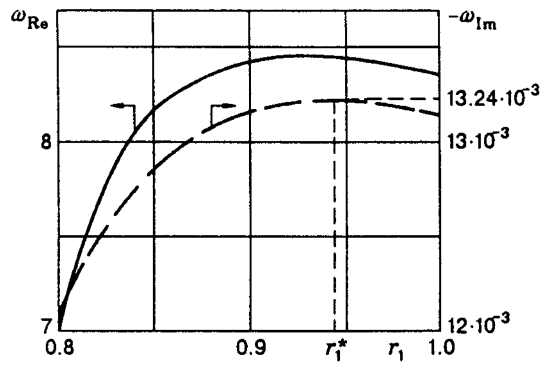

Fig. 2

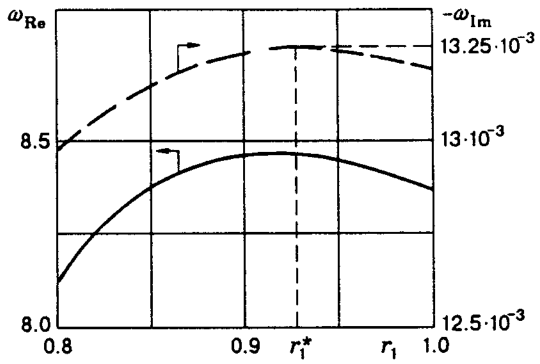

Fig. 3

$$b_i(r_{i-1}) = -\frac{æ_i^2}{r_{i-1}^2}\cos(æ_i r_{i-1}) - \frac{æ_i^2}{r_{i-1}}\sin(æ_i r_{i-1}), \tag{2.1}$$

$$c_i(r_{i-1}) = \left[\frac{4\mu_i æ_i}{r_{i-1}}\cos(æ_i r_{i-1}) - \left(\frac{4\mu_i}{r_{i-1}^2} - æ_i^2(\lambda_i + 2\mu_i)\right)\sin(æ_i r_{i-1})\right]\frac{æ_i}{r_{i-1}},$$

$$d_i(r_{i-1}) = \left[\frac{4\mu_i æ_i}{r_{i-1}}\sin(æ_i r_{i-1}) + \left(\frac{4\mu_i}{r_{i-1}^2} - æ_i^2(\lambda_i + 2\mu_i)\right)\cos(æ_i r_{i-1})\right]\frac{æ_i}{r_{i-1}};$$

$$\Lambda_i = \Delta_i \begin{pmatrix} a_i(r_i) & ib_i(r_i) \\ c_i(r_i) & id_i(r_i) \end{pmatrix} \begin{pmatrix} -d_i(r_{i-1}) & b_i(r_{i-1}) \\ -ic_i(r_{i-1}) & ia_i(r_{i-1}) \end{pmatrix} \quad (i = \overline{1, N}). \tag{2.2}$$

Then

$$\begin{pmatrix} u_N(r_N) \\ \sigma_N(r_N) \end{pmatrix} = \Lambda_N \Lambda_{N-1} * \ldots * \Lambda_1 \begin{pmatrix} u_1(r_0) \\ \sigma_1(r_0) \end{pmatrix} \tag{2.3}$$

or

$$\begin{pmatrix} u_N(r_N) \\ \sigma_N(r_N) \end{pmatrix} = G_N \begin{pmatrix} u_1(r_0) \\ \sigma_1(r_0) \end{pmatrix} \quad \left[G_N = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \Lambda_N \Lambda_{N-1} * \ldots * \Lambda_1\right]. \tag{2.4}$$

In this case, the boundary conditions

$$\sigma_N(r_N) = \sigma_1(r_0) = 0, \tag{2.5}$$

which correspond to conditions (1.4) with $r_0 = R_1$ and $r_N = R_2$, are satisfied.

We shall use these solutions to analyze the behavior of viscoelastic structures. The analysis is aimed at demonstrating the possibility of controlling the damping characteristics of these structures by the methods applied to the synthesis of elastic structures.

Let us first consider a sphere formed from two materials with the same viscoelastic parameters ($\alpha = 0.5$, $\beta = 0.05$, and $A = 0.01$) but with different elastic characteristics: the density and the Lamé parameters of the first (inner) layer correspond to material No. 3 in Table 1, and those for the second (outer) layer correspond to material No. 1. The inner and outer radii of the sphere are $r_0 = 0.8$ m and $r_N = 1.0$ m, respectively. Figure 2 shows the real (ω_{Re}) and imaginary (ω_{Im}) parts of the natural frequency ω versus the coordinate of the viscoelastic-layer interface r_1 (the designations are the same as in Fig. 1). The maximum damping, i.e., a maximum of $\omega_{\mathrm{Im}} = -13.24 \cdot 10^{-3}$, is ensured by a two-layer structure with $r_1^* = 0.944$ m. The same damping is ensured by a two-layer structure whose outer layer is formed from material No. 2. In this case, the coordinate of the interface between the inner (formed from material No. 3) and outer layers is $r_1^* = 0.928$ (Fig. 3).

Figure 4 shows the ω_{Re} and ω_{Im} dependence on r_1 when the density and the Lamé parameters of the first layer correspond to material No. 1, and those for the second layer correspond to material No. 3. In this case,

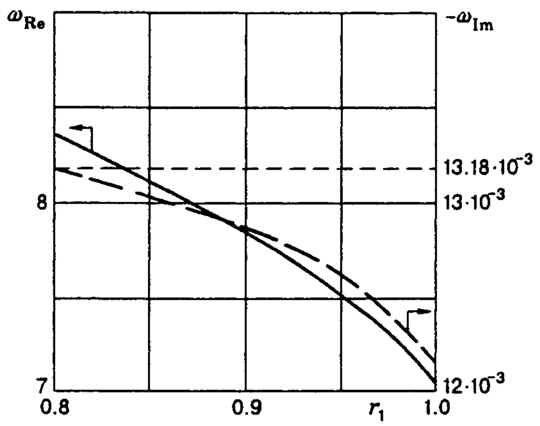

Fig. 4

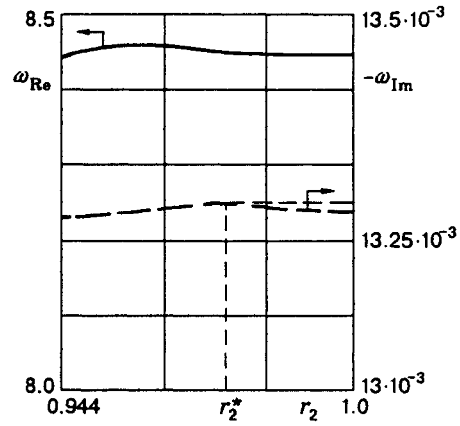

Fig. 5

the maximum damping is ensured by a single-layered sphere formed from material No. 3 ($\omega_{Im} = -13.18\cdot10^{-3}$).

The results shown in Figs. 2–4 indicate that the variation in the thickness and in the order of arrangement of viscoelastic layers permits one to control the damping properties, the maximum damping being ensured by the two-layer structure. Almost the same damping ($\omega_{Im} = -13.25 \cdot 10^{-3}$) is ensured by the three-layer structure: the layer $r_0 = 0.8$ and $r_1 = 0.944$ is filled with material No. 3, the next layer is filled with material No. 2 up to $r_2 = 0.976$, and the outer layer is filled with material No. 1 up to $r_N = 1.0$ (Fig. 5).

The main conclusion that can be drawn on the basis of the results obtained consists in the positive answer to the questions that we put at the beginning of the paper: multilayered viscoelastic structures ensure better damping of natural vibrations than single-layered structures. Certainly, this conclusion is valid for the adopted values of the rheologic parameters and for the adopted type of the relaxation kernel. Thus, subsequent studies can be performed along two lines:

(1) obtaining *a priori* evaluations of the expediency of formulation of the synthesis problem of multilayered structures from a finite set of viscoelastic materials in terms of rheological characteristics;

(2) solution of the synthesis problem when all parameters determining the arrangement of a structure (physical properties of layers materials, the thicknesses and total number of layers) vary.

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